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Radiative Equilibrium: The Effect of a Strong Absorption Line. By E. A. Milne, B.A.

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The approximate form of the equations of transfer of radiant energy when for simplicity the flow of radiation is taken to be "linear" or "tubular" is

$$\frac{dI_\lambda}{dx} = \kappa_\lambda \rho (I_\lambda - B_\lambda), \qquad \frac{dI_\lambda'}{dx} = \kappa_\lambda \rho (B_\lambda - I_\lambda') \quad . \quad (1)$$

$$2\int_0^\infty \kappa_\lambda B_\lambda d\lambda = \int_0^\infty \kappa_\lambda (I_\lambda + I_\lambda') d\lambda \quad . \quad . \quad (2)$$

where I_λ and I_λ' are the outward and inward intensities at the point x, B_λ is the intensity of black-body radiation at the temperature at the point x, and κ_λ is a mean coefficient of absorption, between λ and $\lambda + d\lambda$, for radiation crossing a plane perpendicular to the direction of x; κ_λ is related to the ordinary coefficient k_λ by the approximate relation $\kappa_\lambda = 2k_\lambda$.

The solution of these equations when κ_λ is an arbitrary function of λ would be difficult; the attempt to solve them would hardly be profitable, and their discussion is usually confined to the case where κ_λ is independent of λ. In this note it is proposed to consider briefly the case of a spectrum containing a single absorption line, in the neighbourhood of λ_1, due to a large absorption coefficient κ_{λ_1}; for all other values of λ the absorption is supposed to be constant, given by $\kappa_\lambda = \kappa$.

Now the presence of the dark line in the spectrum means that each layer of gas absorbs much more radiation of wave-length λ_1 than it emits. But in radiative equilibrium the total energy absorbed, of all wave-lengths, is equal to that emitted. Consequently the radiation which the gas fails to emit in wave-length λ_1 must be emitted as extra radiation in the rest of the spectrum, and thus the general distribution of energy in the spectrum is altered. In fact, the total flow of energy being constant, the energy dammed back in the region of the Fraunhofer line must break through elsewhere. The course of events by which this occurs is easily seen. When there is no selective absorption the temperature gradient is, roughly speaking, proportional to the coefficient of absorption κ and to the total flow of energy; when there is selective absorption in the region of $\lambda_1 (\kappa_{\lambda_1} > \kappa)$ the existing temperature gradient* is not sufficient to drive the amount of energy appropriate to λ_1 against the increased resistance given by κ_{λ_1}, and hence the energy accumulates, increasing the temperature gradient until a new steady state is obtained in which the excess gradient in the rest of the spectrum compensates for the deficiency in the neighbourhood of λ_1. A further consequence is that the boundary temperature is slightly decreased in virtue of the cooling

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caused by the increased radiation of the boundary layers in the neighbourhood of λ_1. We shall see later that the combined result of these two effects—increased gradient and decreased boundary temperature—is to increase the intensity on the violet side of the maximum, and to decrease or increase the intensity on the red side according as the absorption line is or is not sufficiently far towards the red.

Let the symbols I_λ, I_λ', B_λ denote the solutions of equations (1) and (2) in the case when there is no selective absorption, *i.e.* when $\kappa_\lambda = \kappa$ for all λ; and let i_λ, i_λ', b_λ denote the solutions in the case when there is a single absorption line, $\kappa_\lambda = \kappa (\lambda \neq \lambda_1)$, *the total flux F being the same in the two cases.* Thus

$$F = \int_0^\infty (I_\lambda - I_\lambda') d\lambda = \int_0^\infty (i_\lambda - i_\lambda') d\lambda.$$

Put $\kappa_{\lambda_1}/\kappa = n$, $\kappa \rho dx = d\tau$. Differentiating (2) and using (1) we have *

$$2\frac{db}{dx}\int_0^\infty \kappa_\lambda \frac{db_\lambda}{db} d\lambda = \rho \int_0^\infty \kappa^2{}_\lambda (i_\lambda - i_\lambda') d\lambda \quad . \quad . \quad (3)$$

but

$$\int_0^\infty \kappa_\lambda \frac{db_\lambda}{db} d\lambda = \int_0^\infty \kappa \frac{db_\lambda}{db} d\lambda + \int_{\lambda_1}^{\lambda_1 + d\lambda_1} (\kappa_{\lambda_1} - \kappa) \frac{db_\lambda}{db} d\lambda$$

$$= \kappa \left[1 + (n - 1) \frac{db_{\lambda_1}}{db} d\lambda_1 \right],$$

and similarly

$$\int_0^\infty \kappa^2{}_\lambda (i_\lambda - i_\lambda') d\lambda = \kappa^2 [F + (n^2 - 1)(i_{\lambda_1} - i_{\lambda_1}') d\lambda_1].$$

Hence (3) becomes

$$\frac{db}{d\tau} = \frac{1}{2} \frac{F + (n^2 - 1)(i_{\lambda_1} - i_{\lambda_1}') d\lambda_1}{1 + (n - 1)\frac{db_{\lambda_1}}{db} d\lambda_1} \quad . \quad . \quad . \quad (4)$$

Expanding this (assuming $(n - 1) d\lambda_1$ is small) and putting

$$b = B + \delta B,$$

we find

$$\frac{d(\delta B)}{d\tau} = \frac{1}{2}(n - 1) d\lambda_1 \left[(n + 1)(i_{\lambda_1} - i_{\lambda_1}') - F \frac{dB_{\lambda_1}}{db} \right] \quad . \quad (5)$$

to the first order. This equation determines the *increase in the temperature gradient* due to the absorption line. Further, equation (2) becomes

$$2b - i - i' + (n - 1) d\lambda_1 (2b_{\lambda_1} - i_{\lambda_1} - i_{\lambda_1}') = 0.$$

At the boundary $i = F$, $i' = i_{\lambda_1}' = 0$, $b_0 = B_0 + \delta B_0$, and thus

$$2\delta B_0 = -(n - 1) d\lambda_1 (2B_{\lambda_1} - i_{\lambda_1})_0 \quad . \quad . \quad . \quad (6)$$

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* The symbols i, b, etc., without suffix denote the corresponding integrated radiation of all wave-lengths.

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to the order considered. This equation determines the *decrease in the boundary temperature.*

Except when $\lambda = \lambda_1$, the intensities i_λ and i_λ' will differ only slightly from I_λ and I_λ'. Put then

$$i_\lambda = I_\lambda + \delta I_\lambda, \qquad i_\lambda' = I_\lambda' + \delta I_\lambda' \quad (\lambda \neq \lambda_1).$$

The equation for δI_λ is readily found to be

$$\frac{d(\delta I_\lambda)}{d\tau} = \delta I_\lambda - \delta B_\lambda = \delta I_\lambda - \frac{dB_\lambda}{dB}\delta B \quad . \quad . \quad (7)$$

whence

$$\delta I_\lambda = e^\tau \int_\tau^\infty \frac{dB_\lambda}{dB}\delta B e^{-\tau} d\tau$$

$$= \tfrac{1}{2}(n-1)d\lambda_1 e^\tau \int_\tau^\infty \left[(i_{\lambda_1} - B_{\lambda_1})_0 + \int_0^\tau \left\{ (n+1)(i_{\lambda_1} - i_{\lambda_1}') - 2B_0\frac{dB_{\lambda_1}}{dB} \right\} d\tau \right] \frac{dB_\lambda}{dB} e^{-\tau} d\tau \quad (8)$$

on inserting the value of δB calculated from (5) and (6). To evaluate this we require the values of i_{λ_1} and i_{λ_1}'; it is clearly sufficient, to the order considered, to neglect the change in the temperature distribution in calculating i_{λ_1} and i_{λ_1}'. The equations are

$$\frac{di_{\lambda_1}}{d\tau} = n(i_{\lambda_1} - B_{\lambda_1}), \qquad \frac{di_{\lambda_1}'}{d\tau} = n(B_{\lambda_1} - i_{\lambda_1}'),$$

yielding

$$i_{\lambda_1} = e^{n\tau} \int_\tau^\infty nB_{\lambda_1} e^{-n\tau} d\tau, \qquad i_{\lambda_1}' = e^{-n\tau} \int_0^\tau nB_{\lambda_1} e^{n\tau} d\tau \quad . \quad (9)$$

Using these values in (8) and putting $\tau = 0$, we obtain after some reduction

$$(\delta I_\lambda)_0 = \frac{(n-1)d\lambda_1}{n} \left[\tfrac{1}{2}(n+1) \int_0^\infty (i_{\lambda_1} + i_{\lambda_1}' - 2B_{\lambda_1}) \frac{dB_\lambda}{dB} e^{-\tau} d\tau \right.$$
$$\left. + \int_0^\infty B_{\lambda_1} \frac{dB_\lambda}{dB} e^{-\tau} d\tau - \tfrac{1}{2}(i_{\lambda_1})_0 \frac{(I_\lambda - B_\lambda)_0}{B_0} \right] \quad (10)$$

This gives the redistribution over the rest of the spectrum of the energy missing in $(\lambda_1, \lambda_1 + d\lambda_1)$. It may be verified that the result of integrating this over all wave-lengths is precisely $(I_{\lambda_1} - i_{\lambda_1})_0 d\lambda_1$, as it should be.

Were the intensity in the line itself, $(i_{\lambda_1})_0$, small compared with that of the neighbouring continuous spectrum, $(I_{\lambda_1})_0$, the only appreciable term in (10) would be the second. But however intense the absorption, the intensity in the line itself can never be less than that corresponding to the boundary temperature. This is obvious physically, and from (9) it can be seen that as $n \rightarrow \infty$, $(i_{\lambda_1})_0 \rightarrow (B_{\lambda_1})_0$. Thus in strict radiative equilibrium the intensity of radiation in the darkest possible line cannot fall much below that corresponding to a temperature of about $2^{-\frac{1}{4}}T_1$, T_1

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being the effective temperature. Instead of approximating to (10) as it stands, we shall consider its limit as $n \rightarrow \infty$, *i.e.* we consider the redistribution of energy corresponding to an absorption line as dark as possible.* Detailed analysis which we here omit shows that

$$\lim_{n \to \infty} (\delta I_\lambda)_0 = d\lambda_1 \left[\int_0^\infty B_{\lambda_1} \frac{dB_\lambda}{dB} e^{-\tau} d\tau - \tfrac{1}{2} (B_{\lambda_1})_0 \left\{ \frac{I_\lambda - B_\lambda}{B} + \frac{dB_\lambda}{dB} \right\}_0 \right] \qquad (11)$$

We shall take this as our approximation. Now we know from the usual theory that $(I_\lambda)_0$ is very nearly the radiation corresponding to the temperature given by $B = 2B_0$. Thus approximately

$$(I_\lambda - B_\lambda)_0 = B_0 \left(\frac{dB_\lambda}{dB} \right)_0,$$

and hence approximately from (11)

$$\begin{aligned}(\delta I_\lambda)_0 &= d\lambda_1 \left[\int_0^\infty B_{\lambda_1} \frac{dB_\lambda}{dB} e^{-\tau} d\tau - \left(B_{\lambda_1} \frac{dB_\lambda}{dB} \right)_0 \right] \\ &= d\lambda_1 \int_0^\infty \frac{d}{d\tau} \left(B_{\lambda_1} \frac{dB_\lambda}{dB} \right) e^{-\tau} d\tau \qquad . \quad . \quad . \quad (12)\end{aligned}$$

On substituting the explicit value for the temperature distribution, $B = B_0(1 + \tau)$, and putting $B/B_0 = y$, we have

$$(\delta I_\lambda)_0 = C\lambda^{-5}\lambda_1{}^{-5} d_1\lambda_1 \int_1^\infty \frac{d}{dy} \left\{ \frac{\frac{1}{4} a y^{-\frac{5}{4}} e^{a y^{-\frac{1}{4}}}}{(e^{a_1 y^{-\frac{1}{4}}} - 1)(e^{a y^{-\frac{1}{4}}} - 1)^2} \right\} e^{-y} dy \qquad (13)$$

where we have put

$$a = hc/\lambda R T_0, \qquad a_1 = hc/\lambda_1 R T_0.$$

We know that $\int_0^\infty (\delta I_\lambda)_0 d\lambda$ is positive, being in fact the missing energy. But our expression (13) teaches us the interesting fact that $(\delta I_\lambda)_0$ *may be negative for certain values of* λ *and* λ_1. For on performing the differentiation it appears that so long as $e^{-a y^{-\frac{1}{4}}}$ and $e^{-a_1 y^{-\frac{1}{4}}}$ are small compared with unity, the integrand in (13) is negative when

$$y > \left(\frac{a + a_1}{5} \right)^4 \qquad . \quad . \quad . \quad . \quad (14)$$

The value of (13) depends principally on the values of the integrand for small values of y; by analogy with other cases we may expect it to be equal roughly † to the value of the integrand (without the

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* It must be remembered that $n d\lambda_1$ must always be so small that terms such as $n I_{\lambda_1} d\lambda_1 / I$, etc., are small compared with unity. Thus as $n \rightarrow \infty$, we must suppose that $d\lambda_1 \rightarrow 0$. But our results will hold approximately (for a non-zero $d\lambda_1$) however large n may be, provided the total disturbance due to the line is small. Equation (4) appears to require that $n^2 d\lambda_1$ should be small too, but analysis shows that this is not the case.

† *M.N.*, 81, 384 (March 1921).

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exponential weighting factor) for $y = 2$. We shall thus be fairly safe in saying that $(\delta I_\lambda)_0$ is negative or positive according as

$$a + a_1 \lesseqgtr 5 \times 2^{\frac{1}{4}} = 5{\cdot}9.$$

We know that the position of maximum intensity $(\lambda_{\max})$ is given by $a = 6$ approximately.* If now we express λ and λ_1 as multiples of $\lambda_{\max}$,

$$\lambda = r\lambda_{\max}, \qquad \lambda_1 = r_1\lambda_{\max},$$

these conditions become roughly

$$\frac{1}{r} + \frac{1}{r_1} \lesseqgtr 1 \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If $r_1 < 1$, $(\delta I_\lambda)_0$ will be positive for all values of λ; but if $r_1 > 1$, $(\delta I_\lambda)_0$ will be negative for all wave-lengths to the red side of that given by

$$\lambda = \frac{r_1}{r_1 - 1}\lambda_{\max} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and positive for all wave-lengths to the violet side of this. This wave-length is itself on the red side of the maximum. Thus *the effect of an absorption line which lies on the violet side of the maximum is to brighten the continuous spectrum, slightly, everywhere; but if the absorption line lies to the red side of the maximum, there is a certain wave-length also lying to the red side of the maximum such that on the violet side of this wave-length the spectrum is brightened, on the red side darkened.* The region of darkening, when it occurs, begins at a point which is closer to the position of maximum the further the absorption line is to the red, as is seen from (16). It appears probable from a rough investigation of (13) that the position of maximum brightening is always some way to the violet of the ordinary maximum, and near $a = 8$, and that when darkening occurs in the red due to a line far in the red there is a position of maximum darkening near $a = 4$.

For a number of lines these effects must be superimposed. Thus *in strict radiative equilibrium the presence of intense absorption lines necessitates a redistribution of the missing energy over the rest of the spectrum, the consequent distortion of the spectrum consisting in general of a strengthening of the violet with a possible weakening of the red, with a slight shift of the maximum towards the violet.* It is difficult to estimate the practical importance of this effect in actual stars or in the sun; it is probably slight—a detailed bolograph of the spectrum would be required showing exactly the breadths and troughs of the lines. It is important to recognise, however, that assuming radiative equilibrium the general distribution of energy in the continuous spectrum is liable to be seriously changed if there is considerable selective absorption.

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This investigation applies only to the case in which the lines are due to selective absorption (κ_λ = const.) throughout the whole of that depth of the star from which the "photospheric" light is radiated and not merely to selective absorption in a thin layer near the boundary. Further, it is assumed that in the radiating matter the mechanism of radiation and absorption is adequate for the free transformation, when necessary, of energy absorbed in a particular wave-length into heat energy or into energy of radiation in other wave-lengths.

Solar Physics Observatory, Cambridge:
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The Sun-Spot Group and the Magnetic Disturbances, 1921 *May* 8–21. By the Rev. A. L. Cortie, S.J. (Plate 17.)

The sun-spot which appeared on the sun's E. limb, 1921 May 8, developed into a very large and active group. At first it was a single spot with scattered umbræ in very dark penumbræ. Spots of this type are not unfrequently connected with magnetic disturbance.* It appeared in a region of the sun which had not been disturbed for some time. When it appeared on the sun's E. limb it would have been about five days old. By May 13 it had separated into two large spots. Their maximum area 16·5, in units 1/5000 of the sun's visible disc, was attained on May 14. The leader spot was a composite spot containing two umbræ, the mean heliographic co-ordinates of which were latitude + 1·4° and longitude 6·4°. The following spot was a large single spot, latitude + 0·2°, longitude 358·8°. The type was II$_b$.† The whole group extended over about 12° in longitude, and 6° in latitude, so that a very considerable area of the sun's surface was disturbed.

Not only was the spot large and active, but it was also an equatorial spot. Moreover, the mean heliographic latitude of the earth being − 2·4°, it was placed very favourably relatively to the spot-outburst. These are conditions under which a great magnetic storm is most likely to occur.

A detailed description of the chief phases of the series of magnetic disturbances which accompanied the passage of the spot-group across the sun, as recorded at this observatory, was communicated to *Nature*.‡ Briefly, after minor yet well-marked preparatory movements of the needles on May 11–12, the first phase in the greater activity of disturbance began abruptly on May 13 at 13^h 10^m. A second and more intense phase of the storm commenced on May 14, 16^h 8^m. The speed and amplitude of the oscillations of the needles grew gradually greater, until May 15, 0^h 45^m, when the beginning of the greatest intensity of

* *M.N.*, 73, 151.

† *Memoirs B.A.A.*, 23, 2, 79.

‡ *Nature*, 1921 June 2.